Excercise sheet 1

Computational Metaphysics

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This sheet's topics will be discussed in the tutorial session (wednesday 2pm).
Due: Tuesday April 26th, until end of lecture.

Summary of so-far discussed natural deduction rules:

$$\frac{A \quad B}{A \land B}\ conjI \qquad \frac{A \land B}{A}\ conjunct1 \qquad \frac{A \land B}{B}\ conjunct2$$

$$\frac{A}{A \lor B}\ disjI1 \qquad \frac{B}{A \lor B}\ disjI2 \qquad \frac{A \lor B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ disjE$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \longrightarrow B}\ impI \qquad \frac{A \longrightarrow B \quad A}{B}\ mp \qquad \frac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\ notI \qquad \frac{\neg A \quad A}{\bot}\ notE$$

$$\frac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}\ ccontr \qquad \frac{\neg\neg A}{A}\ notnotD \qquad \frac{}{\neg A \lor A}\ excluded\_Middle$$

Excercise 1: Logic formulae.

Before we start using the calculus of natural deduction (ND) for proving logical expressions, we become acquainted with the basic logical connectives, the quantifiers and the remaining components of a logical formula. To that end, please give appropriate formalizations of the following expressions stated in natural language. You may freely choose appropriate names for variables and further identifiers.

(a) "The ship is huge and it is blue."
(b) "I'm sad if the sun does not shine."
(c) "Either it's raining or it is not."
(d) "I'm only going if she is going!"
(e) "Everyone loves chocolate or ice cream."
(f) "There is somebody who loves ice cream and loves chocolate as well."
(g) "Everyone has got someone to play with."
(h) "Nobody has somebody to play with if they are all mean."
(i) "Cats have the same annoying properties as dogs."

Excercise 2: Order of formulae.

Please give the order (propositional, first-order, higher-order) of the following logical formulae. Explain your answers.

(a) $p(a) \lor p(b)$
(b) $\forall X.\forall Y.(\forall P.P(X) \longrightarrow P(Y)) \leftrightarrow X = Y$
(c) $\exists A.p(a) \land \forall B.A \neq B \longrightarrow \neg p(B)$
(d) $p(f(a)) \land \forall Q.\forall R.R(Q) \land Q(f)$

Excercise 3: ND warm-up.

Prove (at least four of) the following statements by giving an explicit proof within the natural deduction calculus. Please make sure that every inference step in your proof is fine-grained and annotated with the respective calculus rule name. The statements are increasingly complex to prove, so don't get frustrated if starting from the bottom.

(a) $A \wedge B \longrightarrow C,\ B \longrightarrow A,\ B \vdash C$

(b) $A \vdash B \longrightarrow A$

(c) $A \longrightarrow (B \longrightarrow C) \vdash B \longrightarrow (A \longrightarrow C)$

(d) $\neg A \vdash A \longrightarrow B$

(e) $\vdash A \vee \neg A$ (you may not use rule *excluded_Middle* here)

(f) $A \vee B,\ \neg A \vdash B$

(g) $\neg A \vee B \vdash A \longrightarrow B$

(h) $\vdash ((A \longrightarrow B) \longrightarrow A) \longrightarrow A$

(i) $A \leftrightarrow B \vdash (A \wedge B) \vee (\neg A \wedge \neg B)$ (**this is a tedious one!**)

Recall that $A \leftrightarrow B$ is defined as $(A \longrightarrow B \wedge B \longrightarrow A)$.

Excercise 4: Relating Hilbert-systems to ND.

So-called *Hilbert* deduction systems do not include as many inference rules as natural deduction calculi do. In fact, a Hilbert system for propositional logic only consists of one inference rule, denoted *modus ponens* (MP)

$$\frac{A \quad A \longrightarrow B}{B}\ MP$$

together with the following four axiom schemes (P1 - P4):

(P1) $A \longrightarrow A$

(P2) $A \longrightarrow (B \longrightarrow A)$

(P3) $(A \longrightarrow (B \longrightarrow C)) \longrightarrow ((A \longrightarrow B) \longrightarrow (A \longrightarrow C))$

(P4) $(\neg A \longrightarrow \neg B) \longrightarrow (B \longrightarrow A)$

Note that Hilbert systems and natural deduction systems use a quite different approach for defining a system of logical deduction: Whereas Hilbert systems have only few deductions rules and a (comparably) large number of axiom schemes, natural deduction systems typically consists of many deduction rules but few to none axiom schemes.

Task. Despite their superficial differences, both systems are equally powerful in the sense that one can give a proof in one system if there exists a proof in the other one and vice versa. Show that each axiom scheme of the above Hilbert system is a tautology in the natural deduction calculus.
For example, in order to show that axiom P1 can be deduced using ND, prove the statement $\vdash A \longrightarrow A$ only using inference rules of the ND calculus (as done in Excercise 3).