Exercise sheet 3

Computational Metaphysics

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The exercises will be discussed in the tutorial session (wednesday 2pm).

Additional natural deduction rules for this exercise sheet:

First-order rules:

$$\frac{A}{\forall x.A}\ allI$$

if x does not occur free in any premises of this subtree

$$\frac{\forall x.A}{A[t/x]}\ allE$$ [a]

$$\frac{\exists x.B \quad \begin{matrix}[B]\\ \vdots \\ A\end{matrix}}{A}\ exE$$

if x does not occur free in A and in any premise of the right subtree except B

$$\frac{A[t/x]}{\exists x.A}\ exI$$

Further handy (derived) rules you may use:

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ A\end{matrix}}{A \leftrightarrow B}\ iffI$$

$$\frac{A \leftrightarrow B \qquad \begin{matrix}[A \longrightarrow B, B \longrightarrow A]\\ \vdots \\ C\end{matrix}}{C}\ iffE$$

[a] Actually, the Isabelle equivalent is a bit more "expressive" (omitted here), you may assume either of them.

Please solve all the following excercises using the ISABELLE *system. Add all your solutions to the same* `.thy` *file, create a* `.pdf` *file and upload both using the KVV system. The* `.thy` *file (and thus the theory) should be named using the format* `Lastname`$_1$`Lastname`$_2$`Ex03.thy` *as in* `MüllerMeierEx03.thy`*. If you are using a temporary account, please also state the account name somewhere in your solution.*

Exercise 1: First-order ND warmup (Tutorial session).

In this task, we will learn the Isabelle syntax for writing proofs containing quantified variables. In our case, these proofs will be ND proofs (as we are only using the rules of the ND calculus), but the general syntax for handling quantified variables in proofs applies also for more advanced usage of Isabelle.

We prove the following theorems (HOL-style syntax):

(a) $((\forall X.\ p\ X) \wedge (\forall X.\ q\ X)) \vdash (\forall X.\ p\ X \wedge q\ X)$

(b) $\forall X.\ p\ X \vdash \exists X.\ p\ X$

(c) $\exists X.\ p\ X \wedge q\ X \vdash \exists X.\ p\ X$

(d) $\vdash ((\forall X.\ p\ X) \wedge (\forall X.\ q\ X)) \leftrightarrow (\forall X.\ p\ X \wedge q\ X)$

Please complete the proofs on your own (in groups of two, of course).

Exercise 2: First-order ND proofs.

Please prove or refute the following first-order formulae using a ND-style proof or by giving a counter example, respectively. You may use the `nitpick` command to find counter models. Counter models can be stated as `(* comment *)` inside the `.thy` file and false `lemma`s can be completed using the `oops` command.

(a) $(\exists X.\forall Y.\ p\ X\ Y) \longrightarrow (\forall Y.\ \exists X.\ p\ X\ Y)$

(b) $(\forall X.\ p\ X \longrightarrow q) \leftrightarrow ((\exists X.\ p\ X) \longrightarrow q)$

(c) $((\forall X.\ p\ X) \vee (\forall X.\ q\ X)) \leftrightarrow (\forall X.\ (p\ X \vee q\ X))$

(d) $((\exists X.\ pX) \vee (\exists X.qX)) \leftrightarrow (\exists X.\ (p\ X \vee q\ X))$

(e) $(\forall X.\ \exists Y.p\ X\ Y) \longrightarrow (\exists Y.\ \forall X.\ p\ X\ Y)$

(f) $(\neg(\forall X.\ p\ X)) \leftrightarrow (\exists X.\ \neg p\ X)$

Exercise 3: *Prettifying* proofs.

Please read chapter 4 (Isar: A Language for Structured Proofs) of the prog-prove.pdf[1] manual, at least until page 46 (end of chapter 4.2). As of chapter 4.2, some important proof patterns for Isabelle proofs are presented, including proof by case distinction and proof by contradiction. Also, a shorter way of proving equivalences and of employing top-level introduction rules is presented.

Use these techniques to make your proofs of Exercise 2 shorter and prettier! The shortest/best/prettiest proofs wins a prize!

Please do not replace your original proof of Excercise 2 by the one produced here, but rather append a new version (or multiple versions) of it in the document. Note that you may still only use `rule` as a tactic (after the `by` keyword) as we are doing ND proofs. We will start using this kind of proof formalization intensively in the next weeks, even when we leave the low-level proofs of the ND calculus.

Exercise 4: A Riddle.

The following riddle is valid in classical logic: *"There is someone in the pub such that, if he is drinking, then everyone in the pub is drinking."*

This statement is formalized by $\exists X.\ (d\ X \longrightarrow (\forall Y.\ d\ Y))$, where X is an arbitrary object and d is a predicate. Please first give a pen-and-paper proof sketch (use proof by case distinction), then formalize it as a ND-style proof in Isabelle. Here, "ND-style" means that the only allowed proof tactic is `rule` (as always for ND proofs, together with the ND inference names) and `cases` (for the Isar-style case distinction, as learned in Ex. 3 from the prog-prove manual). You may use the following lemma

```
lemma allDeMorgan:"¬ (∀ X. p X) ⟹ (∃ X. ¬ (p X))" by simp
```

for this proof, if necessary.

[1] See `https://isabelle.in.tum.de/dist/Isabelle2016/doc/prog-prove.pdf`