Exercise sheet 5

Computational Metaphysics

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The exercises will be discussed in the tutorial session (wednesday 2pm).

Please solve all the following excercises using the Isabelle system. Add allyour solutions to the same `.thy` file, create a `.pdf` file and upload both using the KVV system. The `.thy` file (andthus the theory) should benamed using the format $\texttt{Lastname}_1\texttt{Lastname}_2\texttt{Ex05.thy}$ *as in* `MüllerMeierEx05.thy`*.If you are using a temporary account, please also state the account name somewhere in yoursolution.* **You may use all proof tactics except for `smt` and `metis` for solving this exercise sheet!**

Exercise 1: Unary Boolean Operators.

In classical logic the number of Boolean truth values is limited to two, consisting of *True* and *False*. In reverse, a function f cannot take many different forms. In this exercise we want to prove the following theorem.

theorem *ex1*:
fixes $f :: bool \Rightarrow bool$
shows $f\ (f\ (f\ n)) = f\ n$

Hint: To prove this theorem, a case (nested) distinction over n, $f\ n$ and/or $f\ (f\ n)$ might be appropriate.

Exercise 2: Leibniz Equality.

In higher-order logic (HOL), we do not need to include equality as a primitive of the language (as opposed to FOL). Instead, we can define a notion of equality. A popular notion of equality is given by Leibniz' characterization of equals: Two objects are equal if and only if they share all properties. This can be expressed in HOL as $\forall P.\ P\ a \longleftrightarrow P\ b$ where a and b are two arbitrary objects.

We define Leibniz equality as follows:

abbreviation *leibnizEq* $:: {'a} \Rightarrow {'a} \Rightarrow bool$ (**infixl** $=^L$ *42*) **where**
$a =^L b \equiv \forall P.\ P\ a \longrightarrow P\ b$

Show that the above definition of *op* $=^L$ using only the implication *op* $\longrightarrow$ is indeed enough to characterize Leibniz equality in HOL. To that end, show:

lemma $a =^L b \Longrightarrow \forall P.\ P\ a \longleftrightarrow P\ b$

Show that if two objects are leibniz-equal, then they are equal (in the standard sense).

lemma $a =^L b \Longrightarrow a = b$

Show that if two objects are equal, then they are leibniz-equal.

lemma $a = b \Longrightarrow a =^L b$

Hence, we can indeed use Leibniz equality as substitute for primitive equality. Nevertheless, we do not really want that. We will discuss reasons for that in the tutorial session.

Exercise 3: To Mock a Mockingbird.

This puzzle was taken from Raymond Smullyans book "To Mock a Mockingbird". It is the first puzzle of chapter 9 which introduces us to a magical forest full of talking birds.

typedecl *bird*

If you meet any bird A in this forest, you might call out the name of a bird B to A. Then bird A will answer with the name of a bird $A \cdot B$."

consts *call* :: *bird* ⇒ *bird* ⇒ *bird* (**infixr** · *51*)

Among the many fantastic birds in the forest some birds are called mockingbirds. When you call out the name of any bird x to them, they will answer with the same answer as x would answer to itself.

definition *mockingbird* **where** $mockingbird\ M \equiv \forall x.\ M \cdot x = x \cdot x$

Some birds are said to compose with other birds:

definition *composes-with* **where** $composes\text{-}with\ C\ A\ B \equiv \forall x.\ A \cdot (B \cdot x) = C \cdot x$

The puzzle

Now as you call out names to birds, it might happen that some bird will answer right back with the name you just said. Such a bird is said to be *fond* of this bird. More formally:

definition *is-fond* **where** $is\text{-}fond\ A\ B \equiv A \cdot B = B$

Furthermore, assume that the forest satisfies two conditions: First, for any two birds, there is a bird they compose with. Second, among the birds in the forest there is a mockingbird.

axiomatization where
C1: $\exists\ C.\ composes\text{-}with\ C\ A\ B$ **and**
C2: $\exists\ M.\ mockingbird\ M$

One rumor has it that every bird is fond of at least one bird.

theorem *first-rumor*: $\forall\ x.\ \exists\ y.\ is\text{-}fond\ x\ y$

Another rumor has it that there is at least one bird that is not fond of any bird.

theorem *second-rumor*: $\exists\ x.\ \forall\ y.\ \neg\ (is\text{-}fond\ x\ y)$

One of the rumors is true, which one? (give a proof)

To solve this puzzle you might have to work with the definitions and equalities. The following example shows how to do this.

```
lemma
  assumes a1: mockingbird M
  assumes a2: A · (M · x) = B
  shows A · (x · x) = B
proof -
  from a1 have ∀x. M · x = x · x unfolding mockingbird-def .
  then have mock: M · x = x · x by (rule allE)
  from mock a2 show ?thesis by (rule subst)
qed
```