Exercise sheet 7

Computational Metaphysics

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The exercises will be discussed in the tutorial session (wednesday 2pm).

Please solve all the following excercises using the Isabelle system. Add allyour solutions to the same `.thy` *file, create a* `.pdf` *file and upload both using the KVV system. The* `.thy` *file (andthus the theory) should benamed using the format* $\texttt{Lastname}_1\texttt{Lastname}_2\texttt{Ex06.thy}$ *as in* `MüllerMeierEx06.thy`*.If you are using a temporary account, please also state the account name somewhere in yoursolution.* **You may use all proof tactics except for `smt` for solving this exercise sheet!**

Exercise 1: *Cantor's Theorem*

The Cantor's theorem states, that the powerset is strictly larger, than the original set. Where the idea seems obvious, the proof requires handeling with instatiation of higher-order variables and a diagonalisation argument.

For automated theorem provers Cantor's theorem is often seen as a first obstacle on the way to a fully functioning higher-order prover. Today we want to test, if you are confident enough to be higher-order theorem provers.

(a) The first formulation of this theorem states, that there exists no surjective function from the original set to its powerset.

theorem *surj-theorem* : $\neg surj(f :: 'a \Rightarrow 'a\ set)$

For this standard problem, you can read the Isabelle tutorial "prog-prove" chapter 4. Come up with a proof you are confident to explain in the tutorial session. (If you are confident in your proving skills, try to come up with the prove yourself).

(b) Now try to formalize a similar argument for the injective cantor theorem.

theorem *inj-cantor* : $\neg inj(f :: 'a\ set \Rightarrow 'a)$

Try to find the proof yourself. If you get stuck, feel free to ask any of the tutors for a tip on the diagonalization argument/set.

Exercise 2: *Leibnitz Equality II: Andrews Equality*

In the lecture we have seen a new definition for equality.

abbreviation *andrews-equality* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infixl** $=^A$ *43*) **where**
$a =^A b \equiv \forall q.\ (\forall z.\ q\ z\ z) \longrightarrow q\ a\ b$

Of course we want to show, that this notion is compatible with our definition for Leibnitz Equality.

abbreviation *leibnizEq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infixl** $=^L$ *42*) **where**
$a =^L b \equiv \forall P.\ P\ a \longrightarrow P\ b$

To this end show

(a) **theorem** $a =^L b \Longrightarrow a =^A b$

(b) **theorem** $a =^A b \Longrightarrow a =^L b$

Exercise 3: ***Boolos: A curious inference***

Remember the motivating example of Boolos from the lectures. We should now be able to formalize the proof in Isabelle. In this exercise you are not allowed to use the induction principle. Therefore we introduce the numbers as described in the lecture.

typedecl i

consts
$one :: i$
$s :: i \Rightarrow i$

theorem *boolos*:
fixes $f :: i \Rightarrow i \Rightarrow i$ **and**
$D :: i \Rightarrow bool$
assumes $\forall n.\ f\ n\ one = s\ one$ **and**
$\forall x.\ f\ one\ (s\ x) = s\ (s\ (f\ one\ x))$ **and**
$\forall n.\ \forall x.\ f\ (s\ n)\ (s\ x) = f\ n\ (f\ (s\ n)\ x)$ **and**
$D\ one$ **and**
$\forall x.\ D\ x \longrightarrow D\ (s\ x)$
shows $D\ (f\ (s\ (s\ (s\ (s\ \ one))))\ (s\ (s\ (s\ (s\ \ one)))))$

(a) Follow the steps of the prove in the slides and formalize the proof in Isabelle. Try to stick to the proof as close as possible.

(b) Optional: Test the strength of the Isabelle tools. Try to shorten the proof as much as possible (without using Induction). The shortest version will win *a price*.

[1]

[1] If you want to have more informations on the proof, you can read the paper "The Curious Inference of Boolos in Mizar and OMEGA" http://mizar.org/trybulec65/20.pdf.